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COMPOSITES THEORY AND PRACTICE

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DISPERSION RELATIONS FOR COMPOSITE STRUCTURES. PART I. BASIC ASSUMPTIONS AND REALTIONSHIPS FOR MONOCLINIC LAMINA

Nowadays, the propagation of elastic waves, particularly Lamb waves, is very often used in detecting damages in different kinds of composite materials. These systems are known as structural health monitoring (SHM). However, the phenomenon of Lamb wave propagation is very complex, especially in the case of thin-walled composite structures. Generally, three types of Lamb waves are observed, namely: longitudinal or pressure waves (L), shear vertical (SV) and shear horizontal (SH). The phase and group velocities of the mentioned waves depend on the thickness of the structure and the frequency of the excited signal. This fact makes proper interpretation of the received dynamic response of the structure difficult or even impossible. Therefore, determining the appropriate dispersion curves for different materials is a very important issue. In the present review, the most commonly used analytical approaches for determining dispersion curves in the case of multilayered composite plates are presented. At the very beginning of this work the solution for single isotropic plates is presented. Next, the fundamental assumptions of the theoretical model, which describe the elastic wave propagation phenomenon in multilayered materials, are discussed. In the first part, the relationships describing the elastic wave propagation for single orthotropic lamina are presented. There are two studied cases: namely when the wave front of the elastic wave travels along the principal directions of the material and when the wave front of the elastic wave travels in any arbitrary direction.

Keywords: **Lamb waves, composite materials, anisotropic layer, dispersion curves, phase velocity, group velocity**

RÓWNANIA DYSPERSJI DLA STRUKTUR KOMPOZYTOWYCH. CZĘŚĆ I. PODSTAWOWE ZAŁOŻENIA I RÓWNANIA DLA WARSTWY MONOKLINICZNEJ

Obecnie zjawisko propagacji fal sprężystszych, a w szczególności fal Lamba jest często wykorzystywane przy projektowaniu różnych systemów wykrywania uszkodzeń w wielowarstwowych materiałach kompozytowych. Systemy te są ogólnie znane pod skrótem SHM (Structural Health Monitoring). Jednakże, zjawisko propagacji fal Lamba w kompozytowych konstrukcjach cienkościennych posiada bardzo skomplikowany charakter. W ogólnym przypadku w zależności od płaszczyzny polaryzacji drgań cząstek rozróżniamy trzy rodzaje fal Lamba, a mianowicie: falę podłużną (L) oraz fale poprzeczne spolaryzowane w kierunku pionowym (SV) oraz poziomym (SH). Dodatkowo, każda z wymienionych fal w zależności od grubości materiału oraz częstotliwości generowanego sygnału posiada odpowiednie mody. Mody te propagują się z różną prędkością zarówno fazową, jak i grupową. Zjawisko to znacznie utrudnia interpretację zarejestrowanej dynamicznej odpowiedzi konstrukcji. W pracy szczegółowo opisano najczęściej wykorzystywane analityczne metody wyznaczania krzywych dyspersji. Na początku przedstawiono rozwiązanie dla jednowarstwowej płyty izotropowej. Następnie omówiono podstawowe założenia teoretycznego modelu propagacji fal sprężystych w materiałach wielowarstwowych. W części pierwszej zaprezentowano równania opisujące zjawisko propagacji fal sprężystych w jednowarstwowych płytach o własnościach ortotropowych. Rozważano dwa przypadki, a mianowicie kiedy czoło fal sprężystych porusza się wzdłuż osi głównych materiału oraz kiedy czoło fali porusza się w dowolnym kierunku.

Słowa kluczowe: **fale Lamba, materiały kompozytowe, warstwa anizotropowa, prędkość fazowa, prędkość grupowa**

INTRODUCTION

In the present decade, the number of structures which are made of composite materials is still increasing. It is particularly visible in the aircraft industry where the majority of fuselage and wings are currently made of composite materials. In composite materials the failure process is much more complex in comparison with traditional isotropic materials (steel or aluminum alloys). Matrix cracking, fiber debonding or delamination are observed inside this kind of materials. These damages are very difficult to find at the early stage of production, therefore it is very important from the safety point of view to develop advanced methods of detecting and evaluating different flaws, especially during the normal operation of the structure. In order to cope with this task, the phenomenon of elastic wave propagation is utilized in many cases [1, 2]. It seems that methods based on analysis of the elastic wave propagation and dynamic response of a structure have great potential in practical applications. Hence these methods are still being intensively developed [3, 4].

The earliest works devoted to elastic wave propagation along the free surface of a semi-infinite elastic half-space were done in 1885 by Lord Rayleigh [5]. Ultrasonic guided waves in flat, single isotropic plates were first discovered in 1917 by Lamb [6]. Additionally, in 1926 Love [7] described shear horizontal waves, where the vibrating particles are polarized in the plane of the plate. A comprehensive study of Lamb waves can be found in Victorov [8], Achenbach [9], Graff [10], Rose [11], Royer and Dieulesaint [12] and Giurgiutiu [13]. It is worth noting here that the above-mentioned waves are generally also known as guided waves.

The presented review is devoted to the propagation of elastic waves in thin-walled composite structures, where generally shear horizontal (SH), shear vertical (SV) and pressure (longitudinal) (L) waves are present. The last two waves (SV, L) are known as Lamb waves. All of these waves are highly dispersive. It means that the phase velocity of the traveling wave depends on the frequency. In the case of single isotropic plates, the dispersion curves are described by relatively simple analytical equations. The solution can be found analytically (in the case of SH waves) or by applying an appropriate numerical procedure. However, in composite materials all guided waves are described by a system of coupled equations [13] and obtaining a solution (dispersion curves) is extremely difficult. There are several analytical and numerical methods, nevertheless, none of them can be considered as a universal method. Moreover, some of them are numerically unstable.

The current work is organized in the following way. At the very beginning a description of dispersion phenomenon in the case of single isotropic plates is given. Next, the theoretical assumption associated with the propagation of guided waves in multilayered materials is presented. The theoretical model concerns only straight crested waves traveling in an infinite thin plate. Having discussed these assumptions, the theoretical relationships describing guided wave propagation in composite material are derived. In the next sections, the analytical methods of determining dispersion curves are described, namely: the Transfer Matrix Method (TMM), Global Matrix Method (GMM) and the Stiffness Matrix Method (SMM). Additionally, numerical approaches are also discussed. Most of that section is devoted to the Semi-Analytical Finite Element method (SAFE).

DISPERSION CURVES IN SINGLE ISOTROPIC PLATE

In the case of a single isotropic plate, the dispersion curves for Lamb waves can be determined with the use of Rayleigh-Lamb equations for the symmetric (pressure or longitudinal wave (L)) and antisymmetric (shear vertical wave (SV)) solution [13], namely:

$$\frac{\tan(pd)}{\tan(qd)} = -\frac{\left(\xi^2 - q^2\right)^2}{4\xi^2 pq}, \quad \frac{\tan(pd)}{\tan(qd)} = -\frac{4\xi^2 pq}{\left(\xi^2 - q^2\right)^2} \tag{1}$$

$$p^2 = \frac{\omega^2}{c_L^2} - \xi^2, \quad q^2 = \frac{\omega^2}{c_S^2} - \xi^2. \tag{2}$$

In the above equations, d denotes the half thickness of the plate, ξ is the wave number $\xi = \omega/c$ ($\omega = 2\pi f$ circular frequency [rad/s]) and c_L and c_S are the pressure and shear wave velocity, respectively:

$$c_L = \sqrt{\frac{1-\nu}{(1+\nu)(1-2\nu)}\frac{E}{\rho}}, \quad c_S = \sqrt{\frac{1}{2(1+\nu)}\frac{E}{\rho}} \tag{3}$$

The pressure and shear wave velocities depend on Young's modulus E, Poisson's ratio ν and density of the material ρ. The above equations are obtained with use of the Helmholtz decomposition. The solution of transcendental equations (1) is not easy. For the arbitrarily chosen values of ω, phase velocity c is sought. Exemplary solutions obtained for aluminum alloy Pa38 (E = 70 GPa, ν = 0.33, ρ = 2700 kg/m³) are shown in Figure 1. Next, the group velocities should be evaluated. In order to cope with it, a simple formula can be used, namely:

$$c_g = c^2\left(c - fd\frac{\partial c}{\partial(fd)}\right) \tag{4}$$

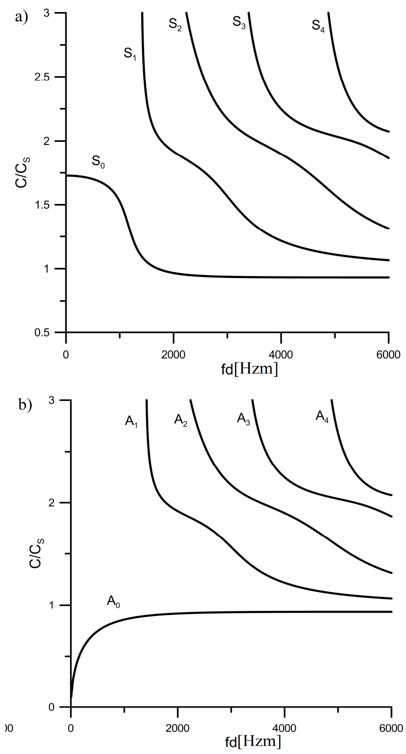

Fig. 1. Wave speed dispersion curves for symmetric (a) and antisymmetric (b) Lamb waves

Rys. 1. Krzywe dyspersji dla symetrycznych (a) i antysymetrycznych (b) modów fal Lamba

The derivative in the above equation can be computed with the used finite difference formula. The corresponding group velocities are presented in Figure 2. In the case of shear horizontal waves (SH), the dispersion curves for the symmetric (5a) and antisymmetric (5b) modes are described by simple analytical relationships, namely:

$$c_{SH}^{S} = \frac{c_s}{\sqrt{1-k^2\left(\dfrac{c_s}{\omega d}\right)^2}}, \quad k = 0, \pi, 2\pi, ... \qquad (5a)$$

$$c_{SH}^{A} = \frac{c_s}{\sqrt{1-k^2\left(\dfrac{c_s}{\omega d}\right)^2}}, \quad k = \frac{\pi}{2}, 3\frac{\pi}{2}, 5\frac{\pi}{2}, ... \qquad (5b)$$

The group velocities of the SH modes can be determined with the use of relation (4).

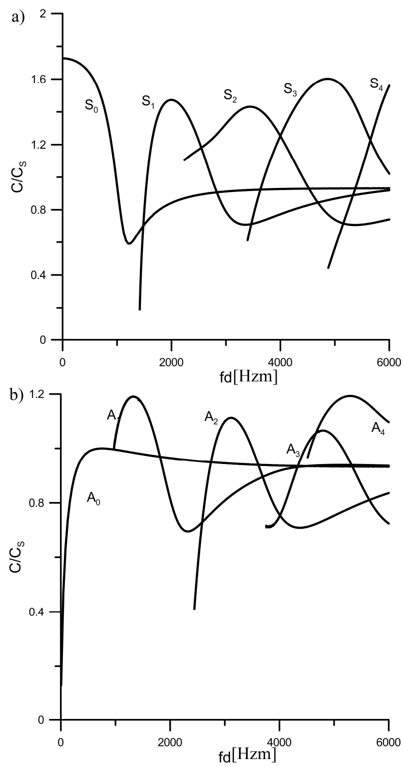

Fig. 2. Group velocity dispersion curves for symmetric (a) and antisymmetric (b) Lamb wave modes

Rys. 2. Krzywe dyspersji prędkości grupowych dla symetrycznych (a) i antysymetrycznych (b) modów fal Lamba

THEORETICAL MODEL OF LAMB WAVES PROPAGATION IN MULTILAYERED MATERIALS - MAIN ASSUMPTIONS

Let us consider the composite layered material which is shown in Figure 3. It is assumed that the analyzed medium consists of n orthotropic layers [13-16]. The mechanical properties of each layer are described in the local coordinate system (x'_1, x'_2, x'_3). It is worth stressing here that the origin of the local coordinate system is chosen to coincide with the top surface of the particular layer, which is shown in Figure 4.

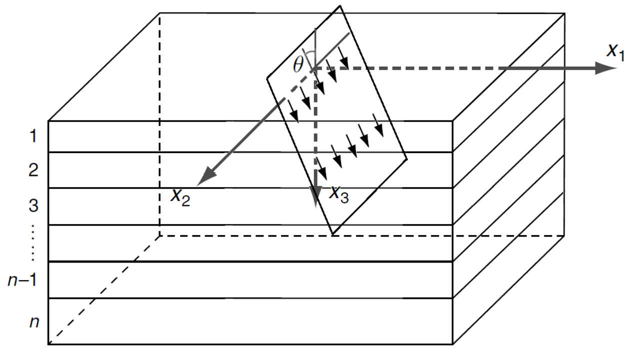

Fig. 3. Composite material with plane wave propagating in x_1 - x_3 direction [13]

Rys. 3. Materiał kompozytowy z falą płaską przemieszczającą się w kierunku x_1 - x_3 [13]

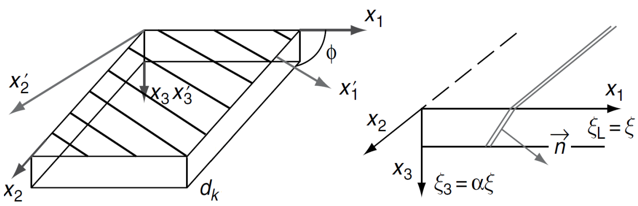

Fig. 4. k-th layer of thickness d_k with local and global coordinate system [13]

Rys. 4. Warstwa k-ta o grubości d_k z lokalnym i globalnym układem współrzędnych [13]

The thickness of the k-th layer is equal to d_k. The particular layer is stacked normally to the x_3 axis of the global coordinate system. Thus, the plane of each layer is parallel to the (x_1, x_2) one of the global coordinate system. The wave is allowed to travel on arbitrary incident angle θ, which is measured with respect to the direction normal to the (x_1, x_2) plane, and along any angle φ. Angle φ is shown in Figure 4. The theoretical model is formulated according to the following assumptions:

1. All the layers are perfectly bonded at their interfaces.
2. The wave propagates along the x_1 - x_3 direction of the global coordinate system. Hence, the mechanical properties of each layer, which are defined in the local coordinate system, have to be transformed to the global coordinate system.
3. In each monoclinic layer [17] there are six partial waves, namely (+L,–L), (+SV,–SV) and (+SH,–SH) representing quasi-longitudinal, quasi-shear vertical and quasi-shear horizontal waves, respectively. The waves with a plus sign are arriving from above the interface of a particular layer and the waves with a minus sign are leaving the interface (Fig. 5). Snell's law requires [18] that all the interacting particular waves must share the same frequency ω and spatial properties in the x_1 direction at each interface. It results that in all the equations which de-

scribe the displacement and stress components, there are the same circular frequency ω and k_1 components of the wave vector. k_1 is the projection of the wave vector of the bulk wave onto the interface.

4. The analyzed composite material is surrounded by vacuum [17, 18]. In other words, it is assumed that the traveling wave does not interact with the external environment. Hence, on the top and bottom surface of the composite material the following stress components are equal to 0: $\sigma_{i,3} = 0$, $i = 1,2,3$.

In the case of layered materials, use of the Helmholtz decomposition does not provide the solution for the Lamb wave equations. Therefore, most of the methods for solving the propagation of Lamb waves in an anisotropic medium are based on the partial wave technique. In this approach, the superposition of three upward and downward propagating waves are assumed. Taking under consideration the above assumptions, a formal solution for displacement can be proposed as follows [15]:

$$(u_1, u_2, u_3) = (U_1, U_2, U_3)e^{i\xi(x_1 \sin\theta + \alpha x_3 - ct)}, \tag{6}$$

where u_i are the displacement components, U_i are the u_i amplitudes, ξ denotes the wave number, α is the unknown parameter (its value will be determined later) and c, t are the phase velocity and time, respectively. For the sake of simplicity, in further discussion the θ angle is set to be equal to θ = 90°.

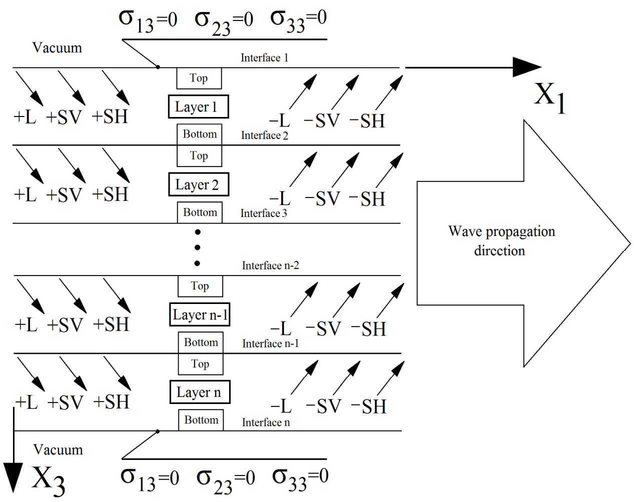

Fig. 5. Monoclinic plate with partial waves [17]

Rys. 5. Monokliniczna płyta z falami cząstkowymi [17]

WAVE PROPAGATION IN SINGLE ORTHOTROPIC LAMINA

In this section two different cases are considered [19, 20], namely guided waves traveling along the principal axes or along any arbitrary direction. In the first case, angle φ = 0° (Fig. 4) and the local (layer) coordinate system are the same as the global coordinate system. In the second case angle $\varphi \neq 0°$ and appropriate transformations of the mechanical properties of the layer from the local to the global coordinate system have to be done. In other words, it is assumed that the x_1 direction of the global coordinate system is the direction of the guided waves propagation.

Wave propagating along principle axes

Let us assume that the mechanical properties of each layer are known and described by the component of the stiffness matrix (transverse isotropy) [21]. Because the local and global coordinate systems are the same in this case, superscript (') will be omitted. The relationships between the stress and strain components can be written as follows:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12.} \end{Bmatrix}. \tag{7}$$

Furthermore the linear relationships between the displacement and strain components are given by [17-19]:

$$\begin{gathered} \varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3} \\ \varepsilon_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}, \quad \varepsilon_{13} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}, \quad \varepsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}. \end{gathered} \tag{8}$$

Finally, the motion equations have the following form:

$$\begin{gathered} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = \rho \frac{\partial^2 u_1}{\partial t^2}, \\ \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} = \rho \frac{\partial^2 u_2}{\partial t^2}, \\ \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = \rho \frac{\partial^2 u_3}{\partial t^2}. \end{gathered} \tag{9}$$

By combining equations (7), (8) and (9), the three following relationships are obtained, namely:

$$C_{11}\frac{\partial^2 u_1}{\partial x_1^2} + C_{55}\frac{\partial^2 u_1}{\partial x_3^2} + (C_{13} + C_{55})\frac{\partial^2 u_3}{\partial x_1 \partial x_3} = \rho \frac{\partial^2 u_1}{\partial t^2} \tag{10a}$$

$$C_{66}\frac{\partial^2 u_2}{\partial x_1^2} + C_{44}\frac{\partial^2 u_2}{\partial x_3^2} = \rho \frac{\partial^2 u_2}{\partial t^2} \tag{10b}$$

$$C_{33}\frac{\partial^2 u_3}{\partial x_3^2} + C_{55}\frac{\partial^2 u_3}{\partial x_1^2} + (C_{13} + C_{55})\frac{\partial^2 u_1}{\partial x_1 \partial x_3} = \rho \frac{\partial^2 u_3}{\partial t^2} \tag{10c}$$

It is worth stressing here that equations (10a) and (10c) are related to the pressure wave (L) and shear vertical wave (SV). Uncoupled relation (10b) is related to the horizontal wave (SH). Moreover, the considered theoretical model is x_2 invariant, thus $\partial/\partial x_2 \equiv 0$.

Solution for SH wave

Substituting displacement component (6) u_2 into (10b), the following equation is obtained:

$$C_{66} + C_{44}\alpha^2 = \rho c^2. \quad (11)$$

The above equation has two roots, namely:

$$\alpha = \pm\sqrt{\frac{\rho c^2 - C_{66}}{C_{44}}}, \quad (12)$$

therefore displacement u_2 and the related stress component σ_{23} can be expressed in the following form:

$$u_2 = U_{21}e^{i\xi(x_1+\alpha x_3-ct)} + U_{22}e^{i\xi(x_1-\alpha x_3-ct)}, \quad (13)$$

$$\sigma_{23} = C_{44}\frac{\partial u_2}{\partial x_3} = i\xi\alpha\left(U_{21}e^{i\xi(x_1+\alpha x_3-ct)} - U_{22}e^{i\xi(x_1-\alpha x_3-ct)}\right), \quad (14)$$

where U_{21}, U_{22} are the unknown amplitudes of partial waves. The above relationships can be rewritten in a matrix form, namely:

$$\begin{Bmatrix} u_2 \\ \sigma_{23} \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ C_{44}\alpha & -C_{44}\alpha \end{bmatrix} \begin{bmatrix} e^{i\xi\alpha x_3} & 0 \\ 0 & e^{-i\xi\alpha x_3} \end{bmatrix} \begin{Bmatrix} U_{21}e^{i\xi(x_1-ct)} \\ U_{22}e^{i\xi(x_1-ct)} \end{Bmatrix}. \quad (15)$$

Solution for L and SV wave

Now, displacements u_1 and u_3 (Eq. (6)) are substituted into equations (10a) and (10c). It results in the following system of linear equations:

$$\begin{bmatrix} C_{11} - \rho c^2 + C_{55}\alpha^2 & (C_{13} + C_{55})\alpha \\ (C_{13} + C_{55})\alpha & C_{55} - \rho c^2 + C_{33}\alpha^2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_3 \end{Bmatrix} = 0. \quad (16)$$

In order to obtain a nontrivial solution, the coefficient matrix determinant has to be equal to zero. To fulfill this condition, the following fourth-order polynomial equation is obtained:

$$A\alpha^4 + B\alpha^2 + C = 0. \quad (17)$$

Generally, there are four real or complex roots of this equation, namely $\alpha_1 = -\alpha_2$ and $\alpha_3 = -\alpha_4$. Now, the displacement and stress components can be written as follows:

$$u_1 = \sum_{j=1}^{4} U_{1j}e^{i\xi(x_1+\alpha_j x_3-ct)}, \quad u_3 = \sum_{j=1}^{4} U_{3j}e^{i\xi(x_1+\alpha_j x_3-ct)}, \quad (18)$$

$$\begin{aligned} \sigma_{33} &= C_{13}\frac{\partial u_1}{\partial x_1} + C_{33}\frac{\partial u_3}{\partial x_3} = \sum_{j=1}^{4} i\xi D_{1j}U_{1j}e^{i\xi(x_1+\alpha_j x_3-ct)}, \\ \sigma_{13} &= C_{13}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) = \sum_{j=1}^{4} i\xi D_{2j}U_{1j}e^{i\xi(x_1+\alpha_j x_3-ct)}, \end{aligned} \quad (19)$$

where:

$$\begin{aligned} D_{1j} &= C_{13} + C_{33}\alpha_j W_j, \quad D_{2j} = C_{55}(\alpha_j + W_j), \\ W_j &= \frac{U_{3j}}{U_{1j}} = \frac{\rho c^2 - C_{11} - C_{55}\alpha_j^2}{(C_{13} + C_{55})\alpha_j}. \end{aligned} \quad (20)$$

As before, by combining relations (18), (19) and (20), the above relationships can be rewritten in the matrix form:

$$\begin{Bmatrix} u_1 \\ u_2 \\ \sigma_{33} \\ \sigma_{13} \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ W_1 & -W_1 & W_3 & -W_3 \\ D_{11} & D_{11} & D_{13} & D_{13} \\ D_{21} & -D_{21} & D_{23} & -D_{23} \end{bmatrix} \begin{bmatrix} e^{i\xi\alpha_1 x_3} & 0 & 0 & 0 \\ 0 & e^{i\xi\alpha_2 x_3} & 0 & 0 \\ 0 & 0 & e^{i\xi\alpha_3 x_3} & 0 \\ 0 & 0 & 0 & e^{i\xi\alpha_4 x_3} \end{bmatrix} \begin{Bmatrix} U_{11}e^{i\xi(x_1-ct)} \\ U_{12}e^{i\xi(x_1-ct)} \\ U_{13}e^{i\xi(x_1-ct)} \\ U_{14}e^{i\xi(x_1-ct)} \end{Bmatrix} \quad (21)$$

Wave propagating along arbitrary direction

Now, it is assumed that the wave can travel in an arbitrary direction. Hence, it is necessary to transform the stiffness matrix of the lamina from the local coordinate system to the coordinate system where the wave propagation direction is determined. After transformation, relationship (7) takes following form [13, 14, 19]:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{Bmatrix}. \quad (22)$$

By combining as before relationships (22), (8), (9), a system of three equations is obtained, namely:

$$\begin{aligned} & C_{11}\frac{\partial^2 u_1}{\partial x_1^2} + C_{66}\frac{\partial^2 u_1}{\partial x_2^2} + C_{55}\frac{\partial^2 u_1}{\partial x_3^2} + 2C_{16}\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \\ & + C_{16}\frac{\partial^2 u_2}{\partial x_1^2} + C_{26}\frac{\partial^2 u_2}{\partial x_2^2} + C_{45}\frac{\partial^2 u_2}{\partial x_3^2} + (C_{12} + C_{66})\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \\ & + (C_{13} + C_{55})\frac{\partial^2 u_3}{\partial x_1 \partial x_3} + (C_{36} + C_{45})\frac{\partial^2 u_3}{\partial x_2 \partial x_3} = \rho\frac{\partial^2 u_1}{\partial t^2} \\ & C_{16}\frac{\partial^2 u_1}{\partial x_1^2} + C_{26}\frac{\partial^2 u_1}{\partial x_2^2} + C_{45}\frac{\partial^2 u_1}{\partial x_3^2} + (C_{12} + C_{66})\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \\ & + C_{66}\frac{\partial^2 u_2}{\partial x_1^2} + C_{22}\frac{\partial^2 u_2}{\partial x_2^2} + C_{44}\frac{\partial^2 u_2}{\partial x_3^2} + +2C_{26}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \\ & + (C_{36} + C_{45})\frac{\partial^2 u_3}{\partial x_1 \partial x_3} + (C_{23} + C_{44})\frac{\partial^2 u_3}{\partial x_2 \partial x_3} = \rho\frac{\partial^2 u_2}{\partial t^2} \\ & (C_{13} + C_{55})\frac{\partial^2 u_1}{\partial x_1 \partial x_3} + (C_{36} + C_{45})\frac{\partial^2 u_1}{\partial x_2 \partial x_3} + \\ & + (C_{36} + C_{45})\frac{\partial^2 u_2}{\partial x_1 \partial x_3} + (C_{23} + C_{44})\frac{\partial^2 u_2}{\partial x_2 \partial x_3} + C_{55}\frac{\partial^2 u_3}{\partial x_1^2} + \\ & + C_{44}\frac{\partial^2 u_3}{\partial x_2^2} + C_{33}\frac{\partial^2 u_3}{\partial x_3^2} + 2C_{45}\frac{\partial^2 u_2}{\partial x_1 \partial x_2} = \rho\frac{\partial^2 u_3}{\partial t^2} \end{aligned} \quad (23)$$

Now, relation (6) is substituted into equation (23). It results in a system of linear equations, namely:

$$[K(\alpha)]\{U\}=\begin{bmatrix} C_{11}-\rho c^2+C_{55}\alpha^2 & C_{16}+C_{45}\alpha^2 & (C_{13}+C_{55})\alpha \\ C_{16}+C_{45}\alpha^2 & C_{66}-\rho c^2+C_{44}\alpha^2 & (C_{36}+C_{45})\alpha \\ (C_{13}+C_{55})\alpha & (C_{36}+C_{45})\alpha & C_{55}-\rho c^2+C_{33}\alpha^2 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}=0 \tag{24}$$

As before, for a nontrivial solution for coefficient matrix determinant (24) has to be equal to zero. It results in a sixth-degree polynomial equation, namely:

$$A\alpha^6+B\alpha^4+C\alpha^2+D=0. \tag{25}$$

There are six real or complex roots of this equation, namely $\alpha_1=-\alpha_2$, $\alpha_3=-\alpha_4$ and $\alpha_5=-\alpha_6$. Now, the displacement and stress components can be written as follows:

$$(u_1,u_2,u_3)=\sum_{j=1}^{6}(1,V_j,W_j)U_{1j}e^{i\xi(x_1+\alpha_j x_3-ct)}, \tag{26}$$

$$(\sigma_{33},\sigma_{13},\sigma_{23})=\sum_{j=1}^{6}i\xi(D_{1j},D_{2j},D_{3j})U_{1j}e^{i\xi(x_1+\alpha_j x_3-ct)} \tag{27}$$

where:

$$\begin{aligned} D_{1j}&=C_{13}+C_{36}V_j+C_{33}W_j\alpha_j, \\ D_{2j}&=C_{55}(\alpha_j+W_j)+C_{45}V_j\alpha_j, \\ D_{3j}&=C_{45}(\alpha_j+W_j)+C_{44}V_j\alpha_j \end{aligned} \tag{28}$$

$$\begin{aligned} V_j&=\frac{U_{2j}}{U_{1j}}=\frac{K_{13}(\alpha_j)K_{23}(\alpha_j)-K_{12}(\alpha_j)K_{33}(\alpha_j)}{K_{22}(\alpha_j)K_{33}(\alpha_j)-K_{23}(\alpha_j)K_{23}(\alpha_j)}, \\ W_j&=\frac{U_{3j}}{U_{1j}}=\frac{K_{12}(\alpha_j)K_{23}(\alpha_j)-K_{13}(\alpha_j)K_{22}(\alpha_j)}{K_{22}(\alpha_j)K_{33}(\alpha_j)-K_{23}(\alpha_j)K_{23}(\alpha_j)} \end{aligned} \tag{29}$$

Relations (26), (27), (28) and (29) can be written in the matrix form:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \sigma_{33} \\ \sigma_{13} \\ \sigma_{23} \end{Bmatrix}=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ V_1 & V_1 & V_3 & V_3 & V_5 & V_5 \\ W_1 & -W_1 & W_3 & -W_3 & W_5 & -W_5 \\ D_{11} & D_{11} & D_{13} & D_{13} & D_{15} & D_{15} \\ D_{21} & -D_{21} & D_{23} & -D_{23} & D_{25} & -D_{25} \\ D_{31} & -D_{31} & D_{33} & -D_{33} & D_{35} & -D_{35} \end{bmatrix} diag\left[e^{i\xi\alpha_j x_3}\right]\begin{Bmatrix} U_{11}e^{i\xi(x_1-ct)} \\ U_{12}e^{i\xi(x_1-ct)} \\ U_{13}e^{i\xi(x_1-ct)} \\ U_{14}e^{i\xi(x_1-ct)} \\ U_{15}e^{i\xi(x_1-ct)} \\ U_{16}e^{i\xi(x_1-ct)} \end{Bmatrix} \tag{30}$$

SUMMARY

At the very beginning of the current review, the solution for isotropic plates is presented. Next, the fundamental assumptions of the theoretical model of elastic waves propagation in multilayered media are discussed. In the following sections, equations describing the propagation of elastic waves in single orthotropic lamina are presented. It is worth stressing here that in the case when the wave front of elastic waves travels along the principal axes of the material (orthotropy direction), determining the dispersion curves simplifies significantly. However, in the case when the wave front travels in an arbitrary direction, obtaining the solution to the problem may be quite difficult mainly due to numerical instabilities.

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